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ANALYTIC FOURIER-FEYNMAN TRANSFORMS AND CONVOLUTION

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ABSTRACT. In this paper we develop an L_p Fourier-Feynman theory for a class of functionals on Wiener space of the form $F(x) = f(\int_0^T \alpha_1 dx, \dots, \int_0^T \alpha_n dx)$. We then define a convolution product for functionals on Wiener space and show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms.

1. INTRODUCTION AND PRELIMINARIES

The concept of an L_1 analytic Fourier-Feynman transform was introduced by Brue in [1]. In [3] Cameron and Storvick introduced an L_2 analytic Fourier-Feynman transform. In [6] Johnson and Skoug developed an L_p analytic Fourier-Feynman transform theory for $1 \le p \le 2$ which extended the results in [1, 3] and gave various relationships between the L_1 and the L_2 theories.

In this paper we first develop an L_p Fourier-Feynman theory for a class of functionals not considered in [1, 3, 6]. We next define a convolution product for functionals on Wiener space and then show that the Fourier-Feynman transform of the convolution product is a product of Fourier-Feynman transforms.

In [3, 6] all of the functionals F on Wiener space and all the real-valued functions F on $\mathbb{R}^n$ were assumed to be Borel measurable. But, as was pointed out in [7, p. 170], the concept of scale-invariant measurability in Wiener space and Lebesque measurability in $\mathbb{R}^n$ is precisely correct for the analytic Fourier-Feynman theory.

Let $C_0[0, T]$ denote Wiener space; that is, the space of real-valued continuous functions x on $[0, T]$ such that $x(0) = 0$. Let $\mathscr{M}$ denote the class of all Wiener measurable subsets of $C_0[0, T]$, and let m denote Wiener measure. $(C_0[0, T], \mathscr{M}, m)$ is a complete measure space and we denote the Wiener integral of a functional F by

$$\int_{C_0[0,T]} F(x) m(dx).$$

A subset E of $C_0[0, T]$ is said to be scale-invariant measurable [4, 7] provided $\rho E \in \mathscr{M}$ for each $\rho > 0$, and a scale-invariant measurable set N is said to be scale-invariant null provided $m(\rho N) = 0$ for each $\rho > 0$. A property

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that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). If two functionals F and G are equal s-a.e., we write $F \approx G$.

Let $\mathbb{C}$, $\mathbb{C}_+$, and $\mathbb{C}_+^\sim$ denote respectively the complex numbers, the complex numbers with positive real part, and the nonzero complex numbers with nonnegative real part. Let F be a $\mathbb{C}$-valued scale-invariant measurable functional on $C_0[0, T]$ such that

$$J(\lambda) = \int_{C_0[0,T]} F(\lambda^{-1/2}x)m(dx)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J^*(\lambda)$ analytic in $\mathbb{C}_+$ such that $J^*(\lambda) = J(\lambda)$ for all $\lambda > 0$, then $J^*(\lambda)$ is defined to be the analytic Wiener integral of F over $C_0[0, T]$ with parameter λ and for $\lambda \in C_+$ we write

$$\int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)m(dx) = J^*(\lambda).$$

Let $q \neq 0$ be a real number, and let F be a functional such that

$$\int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)m(dx)$$

exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral of F with parameter q and we write

$$\int_{C_0[0,T]}^{\text{anf}_q} F(x)m(dx) = \lim_{\lambda\to -iq} \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x)m(dx)$$

where $\lambda \to -iq$ through $\mathbb{C}_+$.

Notation. (i) For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0, T]$ let

$$(T_\lambda(F))(y) = \int_{C_0[0,T]}^{\text{anw}_\lambda} F(x+y)m(dx). \tag{1.1}$$

(ii) Given a number p with $1 \le p \le +\infty$, p and p' will always be related by $1/p + 1/p' = 1$.

(iii) Let $1 < p \le 2$, and let $\{H_n\}$ and H be scale-invariant measurable functionals such that for each $\rho > 0$,

$$\lim_{n\to\infty} \int_{C_0[0,T]} |H_n(\rho y) - H(\rho y)|^{p'} m(dy) = 0. \tag{1.2}$$

Then we write

$$\operatorname*{l.i.m.}_{n\to\infty}(w_s^{p'})(H_n) \approx H \tag{1.3}$$

and we call H the scale invariant limit in the mean of order p'. A similar definition is understood when n is replaced by the continuously varying parameter λ. We are finally ready to state the definition of the L_p analytic Fourier-Feynman transform [6] and our definition of the convolution product.

Definition. Let $q \neq 0$ be a real number. For $1 < p \leq 2$ we define the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ of F by the formula ($\lambda \in \mathbb{C}_+$)

$$(1.4) \qquad (T_q^{(p)}(F))(y) = \underset{\lambda \to -iq}{\mathrm{l.i.m.}}(w_s^{p'})(T_\lambda(F))(y)$$

whenever this limit exists. We define the L_1 analytic Fourier-Feynman transform $T_q^{(1)}(F)$ of F by the formula

$$(1.5) \qquad T_q^{(1)}(F)(y) = \lim_{\lambda \to -iq}(T_\lambda(F))(y)$$

for s-a.e. y. We note that for $1 \leq p \leq 2$, $T_q^{(p)}(F)$ is defined only s-a.e. We also note that if $T_q^{(p)}(F_1)$ exists and if $F_1 \approx F_2$, then $T_q^{(p)}(F_2)$ exists and $T_q^{(p)}(F_2) \approx T_q^{(p)}(F_1)$.

Definition. Let F_1 and F_2 be functionals on $C_0[0, T]$. For $\lambda \in \tilde{\mathbb{C}}_+$ we define their convolution product (if it exists) by

(1.6)
$$(F_1 * F_2)_\lambda(y) = \begin{cases} \displaystyle\int_{C_0[0,T]}^{\mathrm{anw}_\lambda} F_1\left(\frac{y+x}{\sqrt{2}}\right) F_2\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \lambda \in \mathbb{C}_+, \\ \displaystyle\int_{C_0[0,T]}^{\mathrm{anf}_q} F_1\left(\frac{y+x}{\sqrt{2}}\right) F_2\left(\frac{y-x}{\sqrt{2}}\right) m(dx), & \\ & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases}$$

Remark. Our definition of convolution is different than the definition given by Yeh in [9]. For one thing, our convolution product is commutative; that is to say $(F_1 * F_2)_\lambda = (F_2 * F_1)_\lambda$. Next we briefly describe a class of functionals for which we establish the existence of $T_q^{(p)}(F)$. Let n be a positive integer, and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be an orthonormal set of functions in $L_2[0, T]$. For $1 \leq p < \infty$ let $\mathscr{A}_n^{(p)}$ be the space of all functionals F on $C_0[0, T]$ of the form

$$(1.7) \qquad F(x) = f\left(\int_0^T \alpha_1\, dx, \ldots, \int_0^T \alpha_n\, dx\right)$$

s-a.e. where $f : \mathbb{R}^n \to \mathbb{R}$ is in $L_p(\mathbb{R}^n)$ and the integrals $\int_0^T \alpha_j(t)\, dx(t)$ are Paley-Wiener-Zygmund stochastic integrals. Let $\mathscr{A}_n^{(\infty)}$ be the space of all functionals of the form (1.7) with $f \in C_0(\mathbb{R}^n)$, the space of bounded continuous functions on $\mathbb{R}^n$ that vanish at infinity. It is quite easy to see that if F is in $\mathscr{A}_n^{(p)}$, then F is scale-invariant measurable. If $p > 1$ the Feynman integral above should be interpreted as the scale-invariant limit in the mean of the analytic Wiener integral.

2. The transform of functionals in $\mathscr{A}_n^{(P)}$

In this section we show that the L_p analytic Fourier-Feynman transform $T_q^{(p)}(F)$ exists for all F in $\mathscr{A}_n^{(p)}$ and belongs to $\mathscr{A}_n^{(p')}$. We start with some preliminary lemmas.

Lemma 2.1. *Let* $1 \le p \le \infty$, *and let* $F \in \mathcal{A}_n^{(p)}$ *be given by* (1.7). *Then for all* $\lambda \in \mathbb{C}_+$,

$$(T_\lambda(F))(y) = g\left(\lambda; \int_0^T \alpha_1\, dy, \ldots, \int_0^T \alpha_n\, dy\right) \tag{2.1}$$

where

$$\begin{aligned} &g(\lambda; w_1, \ldots, w_n) \\ &\quad \equiv g(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n (u_j - w_j)^2\right\} d\vec{u}. \end{aligned} \tag{2.2}$$

Proof. For $\lambda > 0$, using a well-known Wiener integration theorem we obtain

$$\begin{aligned} (T_\lambda(F))(y) &= \int_{C_0[0,T]} F(\lambda^{-1/2}x + y) m\, dx) \\ &= \int_{C_0[0,T]} f\left(\lambda^{-1/2}\int_0^T \alpha_1\, dx + \int_0^T \alpha_1\, dy, \ldots, \lambda^{-1/2}\right. \\ &\qquad\qquad \left. \times \int_0^T \alpha_n\, dx + \int_0^T \alpha_n\, dy\right) m\,(dx) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f\left(v_1 + \int_0^T dy, \ldots, v_n + \int_0^T \alpha_n\, dy\right) \\ &\qquad\qquad \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n v_j^2\right\} d\vec{v} \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n \left(u_j - \int_0^T \alpha_j\, dy\right)^2\right\} d\vec{u} \\ &= g\left(\lambda; \int_0^T \alpha_1\, dy, \ldots, \int_0^T \alpha_n\, dy\right) \end{aligned}$$

where g is given by (2.2). Now by analytic continuation in λ, (2.1) holds throughout $\mathbb{C}_+$. $\square$

Lemma 2.2. *Let* $F \in \mathcal{A}_n^{(1)}$ *be given by* (1.7), *and let* $g(\lambda; \vec{w})$ *be given by* (2.2). *Then*

(i) $g(\lambda; \cdot) \in C_0(\mathbb{R}^n)$ *for all* $\lambda \in \tilde{\mathbb{C}}_+$;
(ii) $g(\lambda; \vec{w})$ *converges pointwise to* $g(-iq; \vec{w})$ *as* $\lambda \to -iq$ *through* $\mathbb{C}_+$; *and*
(iii) *as elements of* $C_0(\mathbb{R}^n)$, $g(\lambda; \vec{w})$ *converges weakly to* $g(-iq; \vec{w})$ *as* $\lambda \to -iq$ *through values in* $\mathbb{C}_+$.

Proof. We first note that for all $(\lambda, \vec{w}) \in \tilde{\mathbb{C}}_+ \times \mathbb{R}^n$, $|g(\lambda; \vec{w})| \le |\frac{\lambda}{2\pi}|^{n/2}\|f\|_1$. Then (i) follows from a standard argument and the dominated convergence theorem establishes (ii). To establish (iii) let $\mu \in M(\mathbb{R}^n)$, the dual of $C_0(\mathbb{R}^n)$.

By the dominated convergence theorem,

$$\begin{aligned}
&\lim_{\lambda\to -iq}\int_{\mathbb{R}^n} g(\lambda;\vec{w})\,d\mu(\vec{w})\\
&=\lim_{\lambda\to -iq}\int_{\mathbb{R}^n}\left(\frac{\lambda}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} f(\vec{u})\exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n}(u_j-w_j)^2\right\}d\vec{u}\,d\mu(\vec{w})\\
&=\int_{\mathbb{R}^n}\left(\frac{-iq}{2\pi}\right)^{n/2}\int_{\mathbb{R}^n} f(\vec{u})\exp\left\{\frac{iq}{2}\sum_{j=1}^{n}(u_j-w_j)^2\right\}d\vec{u}\,d\mu(\vec{w})\\
&=\int_{\mathbb{R}^n} g(-iq;\vec{w})\,d\mu(\vec{w}).\quad\square
\end{aligned}$$

Our first theorem, which is a direct consequence of Lemma 2.2, shows that the analytic L_1 Fourier-Feynman transform exists for all F in $\mathscr{A}_n^{(1)}$.

Theorem 2.1. *Let $F\in\mathscr{A}_n^{(1)}$ be given by* (1.7). *Then $T_q^{(1)}(F)$ exists for all real $q\neq 0$ and*

$$(T_q^{(1)}(F))(y)\approx g\left(-iq;\int_0^T\alpha_1\,dy,\dots,\int_0^T\alpha_n\,dy\right)\in\mathscr{A}_n^{(\infty)} \tag{2.3}$$

where g is given by (2.2).

Remark. When $1<p\le 2$ and $\operatorname{Re}\lambda=0$, the integral in (2.2) should be interpreted in the mean just as in the theory of the L_p Fourier transform [8].

Theorem 2.2. *Let $1<p\le 2$, and let $F\in\mathscr{A}_n^{(p)}$ be given by* (1.7). *Then the L_p analytic Fourier-Feynman transform of F, $T_q^{(p)}(F)$ exists for all real $q\neq 0$, belongs to $\mathscr{A}_n^{(p')}$ and is given by the formula*

$$(T_q^{(p)}(F))(y)\approx g\left(-iq;\int_0^T\alpha_1\,dy,\dots,\int_0^T\alpha_n\,dy\right) \tag{2.4}$$

where g is given by (2.2).

Proof. We first note that for each $\lambda\in\mathbb{C}_+^\sim$, $g(\lambda;\vec{w})$ is in $L_{p'}(\mathbb{R}^n)$ [5, Lemma 1.1, p. 98]. Furthermore by [5, Lemma 1.2, p. 100]

$$\lim_{\lambda\to -iq}\|g(\lambda;\cdot)-g(-iq;\cdot)\|_{p'}=0. \tag{2.5}$$

Now to show that $T_q^{(p)}(F)$ exists and is given by (2.4) it suffices to show that for each $\rho>0$

$$\lim_{\lambda\to -iq}\int_{C_0[0,T]}\left|g\left(\lambda;\rho\int_0^T\alpha_1\,dy,\dots,\rho\int_0^T\alpha_n\,dy\right)\right.$$
$$\left.-g\left(-iq;\rho\int_0^T\alpha_1\,dy,\dots,\rho\int_0^T\alpha_n\,dy\right)\right|^{p'} m(dy)=0.$$

But

$$\int_{C_0[0,T]}\left|g\left(\lambda;\rho\int_0^T\alpha_1\,dy,\dots,\rho\int_0^T\alpha_n\,dy\right)\right.$$
$$\left.-g\left(-iq;\rho\int_0^T\alpha_1\,dy,\dots,\rho\int_0^T\alpha_n\,dy\right)\right|^{p'} m(dy)$$
$$=\rho^{-n}\int_{\mathbb{R}^n}|g(\lambda;\vec{u})-g(-iq;\vec{u})|^{p'}\exp\left\{-\frac{1}{2\rho^2}\sum_{j=1}^n u_j^2\right\}d\vec{u}$$
$$\le \rho^{-n}\|g(\lambda;\cdot)-g(-iq;\cdot)\|_{p'}^{p'}$$

which goes to zero as $\lambda\to -iq$ by (2.5). Thus $T_q^{(p)}(F)$ exists, belongs to $\mathscr{A}_n^{(p')}$, and is given by (2.4). □

The following example generates an interesting set of functionals belonging to $\mathscr{A}_n^{(p)}$.

Example. Let $1\le p\le+\infty$ be given, and let $\alpha_1,\alpha_2,\dots$ be an orthonormal set of functions from $L_2[0,T]$. Let $F\in L_p(C_0[0,T])$, and for each n define f_n by

$$f_n\left(\int_0^T\alpha_1\,dx,\dots,\int_0^T\alpha_n\,dx\right)\equiv E\left[F(x)\mid\int_0^T\alpha_1\,dx,\dots,\int_0^T\alpha_n\,dx\right].$$

Then, by the definition of conditional expectation, $f_n(\xi_1,\dots,\xi_n)$ is a Borel measurable function, and $\|f_n\|_p\le\|F\|_p$, where

$$\|f_n\|_p=E\left[\left|f_n\left(\int_0^T\alpha_1\,dx,\dots,\int_0^T\alpha_n\,dx\right)\right|^p\right],$$

and

$$\|F\|_p^p=E[|F(x)|^p].$$

Thus $f_n\in\mathscr{A}_n^{(p)}$, and so the analytic Fourier-Feynman transform $T_q^{(p)}(f_n)$ exists for all real $q\ne 0$.

We finish this section by obtaining an inverse transform theorem for F in $\mathscr{A}_n^{(p)}$.

Theorem 2.3. *Let* $1 \le p \le 2$, *and let* $F \in \mathscr{A}_n^{(p)}$. *Let* $q \neq 0$ *be given. Then* (i) *for each* $\rho > 0$,

$$\lim_{\lambda \to -iq} \int_{C_0[0,T]} |T_{\bar\lambda}^{-} T_\lambda(F)(\rho y) - F(\rho y)|^p m(dy) = 0,$$

and (ii) $T_{\bar\lambda}^{-} T_\lambda F \to F$ *s-a.e. as* $\lambda \to -iq$ *through* $\mathbb{C}_+$.

Proof. Proceeding as in the proof of Lemma 2.1, we obtain for all $\lambda \in \mathbb{C}_+$,

$$\begin{aligned}(T_{\bar\lambda}^{-}(T_\lambda(F))(y) &= \left(\frac{\bar\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} g(\lambda; \vec{w}) \exp\left\{-\frac{\bar\lambda}{2}\sum_{j=1}^n \left(w_j - \int_0^T a_j\, dy\right)^2\right\} d\vec{w} \\ &= \left(\frac{\bar\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n (u_j - w_j)^2\right\} \\ &\qquad \times \exp\left\{-\frac{\bar\lambda}{2}\sum_{j=1}^n \left(w_j - \int_0^T \alpha_j\, dy\right)^2\right\} d\vec{u}\, d\vec{w} \\ &= k\left(\lambda, \bar\lambda; \int_0^T \alpha_1\, dy, \dots, \int_0^T \alpha_n\, dy\right)\end{aligned}$$

where $g(\lambda; \vec{w}))$ is given by (2.2) and

$$\begin{aligned} k(\lambda, \bar\lambda; v_1, \dots, v_n) &\equiv k(\lambda, \bar\lambda; \vec{v}) \\ &= \left|\frac{\lambda}{2\pi}\right|^n \int_{\mathbb{R}^{2n}} f(\vec{u}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n (u_j - w_j)^2 - \frac{\bar\lambda}{2}\sum_{j=1}^n (w_j - v_j)^2\right\} d\vec{u}\, d\vec{w}.\end{aligned}$$

But [2, p. 525]

$$\begin{aligned}\int_{\mathbb{R}} \exp\left\{-\frac{\lambda}{2}(u_j - w_j)^2 - \frac{\bar\lambda}{2}(w_j - v_j)^2\right\} dw_j \\ = \left(\frac{\pi}{\operatorname{Re}\lambda}\right)^{1/2} \exp\left\{-\frac{|\lambda|^2}{4\operatorname{Re}\lambda}(u_j - v_j)^2\right\}.\end{aligned}$$

Hence

$$\begin{aligned} k(\lambda, \bar\lambda; \vec{v}) &= \left|\frac{\lambda}{2\pi}\right|^n \int_{\mathbb{R}^n} f(\vec{u}) \left(\frac{\pi}{\operatorname{Re}\lambda}\right)^{n/2} \exp\left\{-\frac{|\lambda|^2}{4\operatorname{Re}\lambda}\sum_{j=1}^n (u_j - v_j)^2\right\} d\vec{u} \\ &= (f * \phi_\varepsilon)(v_1, \dots, v_n)\end{aligned}$$

where

$$\phi(v_1, \dots, v_n) \equiv (2\pi)^{-n/2} \exp\left\{-\frac{1}{2}\sum_{j=1}^n v_j^2\right\}, \qquad \varepsilon \equiv \frac{\sqrt{2\operatorname{Re}\lambda}}{|\lambda|},$$

and

$$\phi_\varepsilon(v_1, \dots, v_n) = \frac{1}{\varepsilon^n} = \frac{1}{\varepsilon^n}\phi\left(\frac{v_1}{\varepsilon}, \dots, \frac{v_n}{\varepsilon}\right).$$

Now

$$\int_{\mathbb{R}^n} \phi(v_1, \dots, v_n)\, dv_1 \cdots dv_n = 1 \quad \text{and} \quad \phi(v_1, \dots, v_n) > 0,$$

so using [8, Theorem 1.18, p. 10] it follows that

$$\lim_{\lambda\to -iq}\int_{\mathbb{R}^n}|k(\lambda,\bar{\lambda};v_1,\dots,v_n)-f(v_1,\dots,v_n)|^p\,d\vec{v}$$

$$(2.6)\qquad =\lim_{\varepsilon\to 0^+}\int_{\mathbb{R}^n}|(f*\phi_\varepsilon)(v_1,\dots,v_n)-f(v_1,\dots,v_n)|^p\,d\vec{v}$$

$$=\lim_{\varepsilon\to 0^+}\|f*\phi_\varepsilon-f\|_p^p=0$$

since $\varepsilon\to 0^+$ as $\lambda\to -iq$ through $\mathbb{C}_+$. But now (i) of the theorem follows easily since for each fixed $\rho>0$,

$$\int_{C_0[0,T]}|T_{\bar{\lambda}}T_\lambda(F)(\rho y)-F(\rho y)|^p m(dy)$$

$$=\rho^{-n}\int_{\mathbb{R}^n}|k(\lambda,\bar{\lambda};\vec{v})-f(\vec{v})|^p\exp\left\{-\frac{1}{2\rho^2}\sum_{j=1}^n v_j^2\right\}d\vec{v}$$

$$\le \rho^{-n}\|f*\phi_\varepsilon-f\|_p^p.$$

Finally, (ii) of the theorem follows since by [8, Theorem 1.25, p. 13] it follows that the function $k(\lambda,\bar{\lambda};v_1,\dots,v_n=(f*\phi_\varepsilon)(v_1,\dots,v_n)$ converges pointwise to the function $f(v_1,\dots,v_n)$ as $\lambda\to -iq$ through $\mathbb{C}_+$. □

Note that in the case $p=2$, $p'=2$, and so for F in $\mathcal{A}_n^{(2)}$, $T_q^{(2)}(F)$ is in $\mathcal{A}_n^{(2)}$ by Theorem 2.2. Hence we have the following theorem.

Theorem 2.4. *Let $F\in\mathcal{A}_n^{(2)}$ be given by (1.7). Then for all real $q\neq 0$,*

$$T_{-q}(T_q(F))\approx F.$$

3. Convolutions and transforms of convolutions

Our first lemma gives an expression for $(F_1*F_2)_\lambda$ for $\lambda\in\mathbb{C}_+$.

Lemma 3.1. *Let $1\le p\le\infty$, and let $F_j\in\bigcup_{1\le p\le\infty}\mathcal{A}_n^{(p)}$ for $j=1,2$ be given by (1.7). Then for all $\lambda\in\mathbb{C}_+$,*

$$(3.1)\qquad (F_1*F_2)_\lambda(y)=h\left(\lambda;\int_0^T\alpha_1\,dy,\dots,\int_0^T\alpha_n\,dy\right)$$

where

(3.2)

$$h(\lambda; w_1, \dots, w_n) \equiv h(\lambda; \vec{w}) = \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(\frac{\vec{w}+\vec{u}}{\sqrt{2}}\right) f_2\left(\frac{\vec{w}-\vec{u}}{\sqrt{2}}\right) \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n} u_j^2\right\} d\vec{u}.$$

Proof. For $\lambda > 0$, using a well-known Wiener integration formula we obtain

$$\begin{aligned}(F_1 * F_2)_\lambda(y) &= \int_{C_0[0,T]} F_1\left(\frac{y+\lambda^{-1/2}x}{\sqrt{2}}\right) F_2\left(\frac{y-\lambda^{-1/2}x}{\sqrt{2}}\right) m(dx) \\ &= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(2^{-1/2}\left[\int_0^T \alpha_1\, dy + u_1\right], \dots, 2^{-1/2}\left[\int_0^T \alpha_n\, dy + u_n\right]\right) \\ &\qquad \times f_2\left(2^{-1/2}\left[\int_0^T \alpha_1\, dy - u_1\right], \dots, 2^{-1/2}\left[\int_0^T \alpha_n\, dy - u_n\right]\right) \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^{n} u_j^2\right\} d\vec{u} \\ &= h\left(\lambda; \int_0^T \alpha_1\, dy, \dots, \int_0^T \alpha_n\, dy\right)\end{aligned}$$

where h is given by (3.2), so (3.1) holds for $\lambda > 0$. Now by analytic continuation in λ, we see that (3.1) holds for all λ in $\mathbb{C}_+$. □

Our next theorem establishes an interesting relationship involving convolutions and analytic Wiener integrals.

Theorem 3.1. *Let* $1 \le p \le \infty$, *and let* $F_j \in \bigcup_{1\le p\le\infty} \mathcal{A}_n^{(p)}$ *for* $j = 1, 2$ *be given by* (1.7). *Then for all* $\lambda \in \mathbb{C}_+$,

(3.3) $$(T_\lambda(F_1 * F_2)_\lambda)(z) = (T_\lambda(F_1))(2^{-1/2}z)(T_\lambda(F_2))(2^{-1/2}z).$$

Proof. It will suffice to establish (3.3) for $\lambda > 0$ since $T_\lambda(F_1 * F_2)_\lambda$, $T_\lambda(F_1)$, and $T_\lambda(F_2)$ all have analytic extensions throughout $\mathbb{C}_+$. So let $\lambda > 0$ be given.

Then by (3.1) and (3.2),

$$
\begin{aligned}
(T_\lambda(F_1 * F_2)_\lambda)(z) &= \int_{C_0[0,T]} (F_1 * F_2)_\lambda(\lambda^{-1/2}x + z)m(dx) \\
&= \int_{C_0[0,T]} h\left(\lambda; \int_0^T \alpha_1 d[\lambda^{-1/2}x + z], \ldots, \right. \\
&\qquad\qquad \left. \int_0^T \alpha_n d[\lambda^{-1/2} + z]\right) m(dx) \\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} h\left(\lambda; v_1 + \int_0^T \alpha_1\, dz, \ldots, v_n + \int_0^T \alpha_n\, dz\right) \\
&\qquad\qquad \times \left\{-\frac{\lambda}{2}\sum_{j=1}^n v_j^2\right\} d\vec{v} \\
&= \left(\frac{\lambda}{2\pi}\right)^{n} \int_{\mathbb{R}^{2n}} f_1\left(2^{-1/2}\left[v_1 + u_1 + \int_0^T \alpha_1\, dz\right], \ldots, \right. \\
&\qquad\qquad \left. 2^{-1/2}\left[v_n + u_n + \int_0^T \alpha_n dz\right]\right) \\
&\quad \times f_2\left(2^{-1/2}\left[v_1 - u_1 + \int_0^T \alpha_1 dz\right], \ldots, \right. \\
&\qquad\qquad \left. 2^{-1/2}\left[v_n - u_n + \int_0^T \alpha_n\, dz\right]\right) \\
&\qquad\qquad \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^\infty [u_j^2 + v_j]\right\} d\vec{u}\, d\vec{v}.
\end{aligned}
$$

Next we make the transformation

$$w_j = 2^{-1/2}(v_j + u_j)$$

and

$$r_j = 2^{-1/2}(v_j - u_j)$$

for $j = 1, 2, \ldots, n$. The Jacobian of this transformation is one and

$$\sum_{j=1}^n [w_j^2 + r_j^2] = \sum_{j=1}^n [u_j^2 + v_j^2].$$

Hence for $\lambda > 0$, using (2.1) and (2.2), we see that

$$
\begin{aligned}
&(T_\lambda(F_1 * F_2)_\lambda)(z)\\
&\quad= \left(\frac{\lambda}{2\pi}\right) \int_{\mathbb{R}^{2n}} f_1\left(w_1 + 2^{-1/2}\int_0^T \alpha_1\,dz, \dots, w_n + 2^{-1/2}\int_0^T \alpha_n\,dz\right)\\
&\qquad\qquad \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n w_j^2\right\}\\
&\qquad \times f_2\left(r_1 + 2^{-1/2}\int_0^T \alpha_1\,dz, \dots, r_n + 2^{-1/2}\int_0^T \alpha_n\,dz\right)\\
&\qquad\qquad \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n r_j^2\right\} d\vec{w}\,d\vec{r}\\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1\left(w_1 + 2^{-1/2}\int_0^T \alpha_1\,dz, \dots, w_n + 2^{-1/2}\int_0^T \alpha_n\,dz\right)\\
&\qquad\qquad \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n w_j^2\right\} d\vec{w}\\
&\qquad \times \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_2\left(r_1 + 2^{-1/2}\int_0^T \alpha_1 dz, \dots, r_n + 2^{-1/2}\int_0^T \alpha_n\,dz\right)\\
&\qquad\qquad \times \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n r_j^2\right\} d\vec{r}\\
&= \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_1(\vec{w}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n \left(w_j - 2^{-1/2}\int_0^T \alpha_j\,dz\right)^2\right\} d\vec{w}\\
&\qquad \times \left(\frac{\lambda}{2\pi}\right)^{n/2} \int_{\mathbb{R}^n} f_2(\vec{r}) \exp\left\{-\frac{\lambda}{2}\sum_{j=1}^n \left(r_j - 2^{-1/2}\int_0^T \alpha_j\,dz\right)^2\right\} d\vec{r}\\
&= (T_\lambda(F_1))(2^{-1/2}z)(T_\lambda(F_2))(2^{-1/2}z).
\end{aligned}
$$

Theorem 3.2. *The following hold for all* $\lambda \in \mathbb{C}_+^\sim$.

(i) *If* $F_1 \in \mathcal{A}_n^{(1)}$ *and* $F_2 \in \mathcal{A}_n^{(1)}$, *then* $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(1)}$.
(ii) *If* $F_1 \in \mathcal{A}_n^{(2)}$ *and* $F_2 \in \mathcal{A}_n^{(2)}$, *then* $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(\infty)}$.
(iii) *If* $F_1 \in \mathcal{A}_n^{(1)}$ *and* $F_2 \in \mathcal{A}_n^{(2)}$, *then* $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(2)}$.
(iv) *If* $F_1 \in \mathcal{A}_n^{(1)}$ *and* $F_2 \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$, *then* $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(1)} \cap \mathcal{A}_n^{(2)}$.
(v) *If* $F_1 \in \mathcal{A}_n^{(1)}$ *and* $F_2 \in \mathcal{A}_n^{(\infty)}$, *then* $(F_1 * F_2)_\lambda \in \mathcal{A}_n^{(\infty)}$.

Proof. (i) Assume F_1 and F_2 belong to $\mathcal{A}_n^{(1)}$ and are given by (1.7). It will suffice to show that $h(\lambda;\cdot)$ given by (3.2) is in $L_1(\mathbb{R}^n)$ for every $\lambda \in \mathbb{C}_+^\sim$. But this follows from the calculations

$$\begin{aligned}
\int_{\mathbb{R}^n} |h(\lambda;\vec{w})|\,d\vec{w} &\le \left|\frac{\lambda}{2\pi}\right|^{n/2} \int_{\mathbb{R}^{2n}} |f_1(2^{-1/2}(\vec{w}+\vec{u}))f_2(2^{-1/2}(\vec{w}-\vec{u}))|\,d\vec{w}\,d\vec{u} \\
&= \left|\frac{\lambda}{2\pi}\right|^{n/2} \int_{\mathbb{R}^n} |f_1(\vec{v})| 2^{n/2} \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w}-\vec{v})|\,d\vec{w}\,d\vec{v} \\
&= \left|\frac{\lambda}{2\pi}\right|^{n/2} ||f_1||_1 ||f_2||_1.
\end{aligned}$$

(ii) In this case for f_1, f_2 in $L_2(\mathbb{R}^n)$ we first note that $h(\lambda;\cdot)$ is in $L_\infty(\mathbb{R}^n)$ since for all $\vec{w} \in \mathbb{R}^n$,

$$\begin{aligned}
|h(\lambda;\vec{w})| &\le \left|\frac{\lambda}{2\pi}\right|^{n/2} \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w}+\vec{u}))||f_2(2^{-1/2}(\vec{w}-\vec{u}))|\,d\vec{u} \\
&\le \left|\frac{\lambda}{2\pi}\right|^{n/2} \left\{\int_{\mathbb{R}^n} |f_1(2^{-1/2}(w+\vec{u}))|^2\,du\right\}^{1/2} \\
&\quad \times \left\{\int_{\mathbb{R}^n} |f_2(2^{-1/2}(\vec{w}-\vec{u}))|^2\,d\vec{u}\right\}^{1/2} \\
&= \left|\frac{\lambda}{2\pi}\right|^{n/2} (\sqrt{2})^n ||f_1||_2 ||f_2||_2 \\
&= \left|\frac{\lambda}{\pi}\right|^{n/2} ||f_1||_2 ||f_2||_2.
\end{aligned}$$

A standard argument now shows that h belongs to $C_0(\mathbb{R}^n)$.

(iii) Let $F_1 \in \mathscr{A}_n^{(1)}$ and $F_2 \in \mathscr{A}_n^{(2)}$ be given by (1.7). It will suffice to show that $h(\lambda;\cdot)$ given by (3.2) is in $L_2(\mathbb{R}^n)$. But this follows from the calculations

$$\begin{aligned}
\int_{\mathbb{R}^n} |h(\lambda;\vec{w})|^2\,d\vec{w} &\le \int_{\mathbb{R}^n} \left|\frac{\lambda}{2\pi}\right|^{n} \left[\int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w}+\vec{u}))f_2(2^{-1/2}(\vec{w}-\vec{u}))|\,d\vec{u}\right. \\
&\quad \left.\times \int_{\mathbb{R}^n} |f_1(2^{-1/2}(\vec{w}+\vec{u}))f_2(2^{-1/2}(\vec{w}-\vec{u}))|\,d\vec{v}\right] d\vec{w} \\
&= \left|\frac{\lambda}{2\pi}\right|^{n} \int_{\mathbb{R}^n} |f_1(\vec{r})| \int_{\mathbb{R}^n} |f_1(\vec{s})| \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w}-\vec{r}) \\
&\qquad \times f_2(\sqrt{2}\vec{w}-\vec{s})|\,d\vec{w}\,d\vec{s}\,d\vec{r} \\
&\le \left|\frac{\lambda}{2\pi}\right|^{n} ||f_1||_1^2 \int_{\mathbb{R}^n} |f_2(\sqrt{2}\vec{w}-\vec{r})|^2\,d\vec{r} \\
&= \left|\frac{\lambda}{2\pi}\right|^{n} (2)^{n/2} ||f_1||_1^2 ||f_2||_2^2.
\end{aligned}$$

Hence $||h||_2 \le |\lambda/\pi\sqrt{2}|^{n/2} ||f_1||_1 ||f_2||_2$.

Finally we note that (iv) follows directly from (i) and (iii) while (v) is immediate. □

In our next theorem we show that the Fourier-Feynman transform of the convolution product is the product of transforms.

Theorem 3.3. (i) *Let* $F_1, F_2 \in \mathscr{A}_n^{(1)}$. *Then for all real* $q \neq 0$,

$$(3.4) \qquad (T_q^{(1)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(1)}(F_2))(2^{-1/2}z).$$

(ii) *Let* $F_1 \in \mathscr{A}_n^{(1)}$ *and* $F_2 \in \mathscr{A}_n^{(2)}$. *Then for all real* $q \neq 0$,

$$(3.5) \qquad (T_q^{(2)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(2)}(F_2))(2^{-1/2}z).$$

(iii) *Let* $F_1 \in \mathscr{A}_n^{(1)}$ *and* $F_2 \in \mathscr{A}_n^{(1)} \cap \mathscr{A}_n^{(2)}$. *Then for all real* $q \neq 0$,

$$(3.6) \qquad (T_q^{(1)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(1)}(F_2))(2^{-1/2}z)$$

and

$$(3.7) \qquad (T_q^{(2)}(F_1 * F_2)_q)(z) = (T_q^{(1)}(F_1))(2^{-1/2}z)(T_q^{(2)}(F_2))(2^{-1/2}z).$$

Proof. Theorem 3.2 together with Theorem 2.2 assures us that all of the transforms on both sides of (3.4) through (3.7) exist. Equations (3.4) through (3.7) now follow from equation (3.3). □

Remark. Throughout this paper, for simplicity we assumed that $\{\alpha_1, \dots, \alpha_n\}$ was an orthonormal set of functions in $L_2[0, T]$. However, all of our results hold provided that $\{\alpha_1, \dots, \alpha_n\}$ is a linearly independent set of functions from $L_2[0, T]$.

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